Quadratic bezier through three points

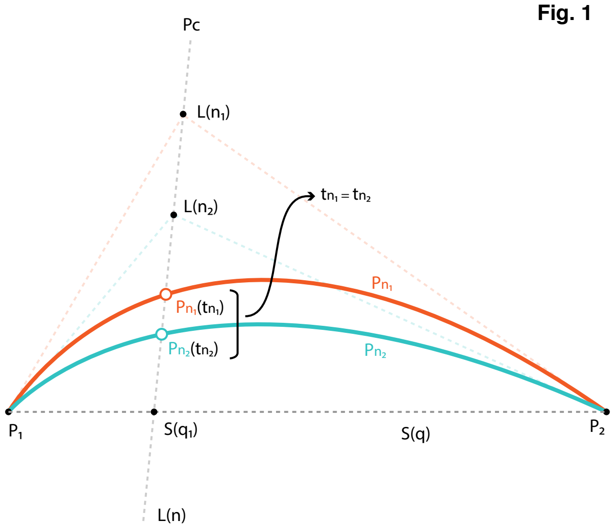
and the “equivalent quadratic bezier (theorem)”

Gabriel Suchowolski, December, 2012

Abstract

As extension of the paper “Quadratic bezier offsetting with selective subdivision – Gabriel Suchowolski, Jul 10, 2012” this document will cover some geometrical properties of the quadratic beziers that I found during my investigation for the paper.

Fig. 1



Equivalent quadratic beziers theorem

All quadratic bezier, that share the start and end control points P_1, P_2, and the “tension point” falling in the rect L, intersect L at the same parametric value of t. (**Fig.1**)

With the rect L defined as

$$L(n) = S(q) + n(P_c - S(q))$$

Where

$$S(q) = P_1 + q(P_2 - P_1) \text{ where } q \in [0,1]$$

and for all n the collection of all quadratic beziers

$$P_n(t) = (1-t)^2 P_1 + 2t(1-t)\,L(n) + t^2 P_2 \quad \text{where } t \in [0,1] \text{ and } n \in \mathbb{R}$$

In other words, (for the same value of q) P_{n_1} intersect L at $P_{n_1}(t_{n_1})$, and P_{n_2} intersect L at $P_{n_2}(t_{n_2})$ where $t_{n_2} = t_{n_1}$

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DEMONSTRATION

Let's see what happen with the intersection of L with a quadratic bezier P_n with an arbitrary point $L(n)$ as "tension point" and the start and end controls points at P_1, P_2, respectively.
Where a point at rect S, $S_q = S(q)$ with $q \in [0,1]$ defines L as

$$L(n) = S_q + n(P_c - S_q)$$

Changing the coordinate system as $S_q \rightarrow (0,0)$, be the translation of $P_n(t)$

$$P_n(t) \rightarrow Q_n(t) = (1-t)^2 Q_1 + 2t(1-t)\,M(n) + t^2 Q_2$$

where (because the change in the coordinate system):

$$\begin{aligned} P_1 &\rightarrow Q_1 \\ S_q &\rightarrow Q_q = (0,0) \\ P_2 &\rightarrow Q_2 \\ P_c &\rightarrow Q_c \\ L(n) &\rightarrow M(n) = nQ_c \end{aligned}$$

The intersection is at another point of M

Then the intersection between $Q_n(t)$ and $M(n)$:

$$(1-t)^2 Q_1 + 2t(1-t)\,M(n) + t^2 Q_2 = M(m)$$

Then $$(1-t)^2 Q_1 + 2t(1-t)\,n\,Q_c + t^2 Q_2 = m\,Q_c$$

Then using the components the system:

$$(1-t)^2\frac{Q_{1x}}{Q_{cx}} + 2t(1-t)\frac{n\cancel{Q_{cx}}}{\cancel{Q_{cx}}} + t^2\frac{Q_{2x}}{Q_{cx}} = m = (1-t)^2\frac{Q_{1y}}{Q_{cy}} + 2t(1-t)\frac{n\cancel{Q_{cy}}}{\cancel{Q_{cy}}} + t^2\frac{Q_{2y}}{Q_{cy}}$$

then

$$(1-t)^2\left[\frac{Q_{1x}}{Q_{cx}} - \frac{Q_{1y}}{Q_{cy}}\right] + \cancel{2t(1-t)(\underset{\overset{||}{0}}{n-n})} + t^2\left[\frac{Q_{2x}}{Q_{cx}} - \frac{Q_{2y}}{Q_{cy}}\right] = 0$$

Note that the roots for t does not depends of m nor n.

So for any value of n all the quadratic beziers $Q_n(t)$ intersect $M(n)$ at the same t

Then for any value of n all quadratic beziers $P_n(t)$ intersect $L(n)$ at the same t

Then, in other words, all quadratic bezier, that share the start and end control points P_1, P_2, and with the "tension point" falling in the rect L intersect L at the same parametric value of t

Lets name this quadratic beziers as "equivalent quadratic beziers"

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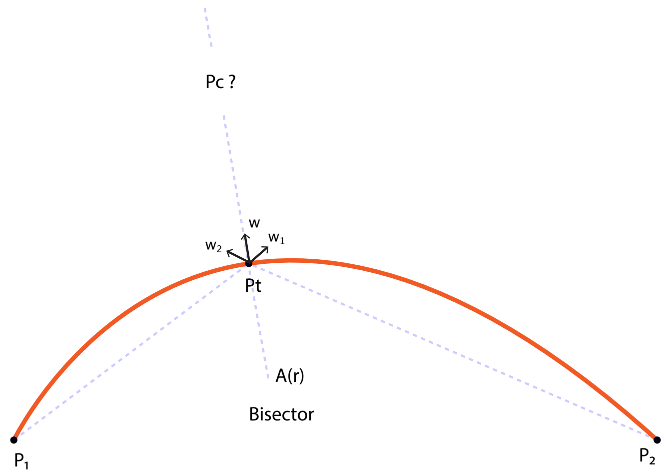
Quadratic bezier through three points with P_c and P_t through the bisector

Fig.. 2



Let be the quadratic bezier

$$P_c(t) = (1-t)^2 P_1 + 2t(1-t)P_c + t^2 P_2$$
$$where \quad t \in [0,1]$$

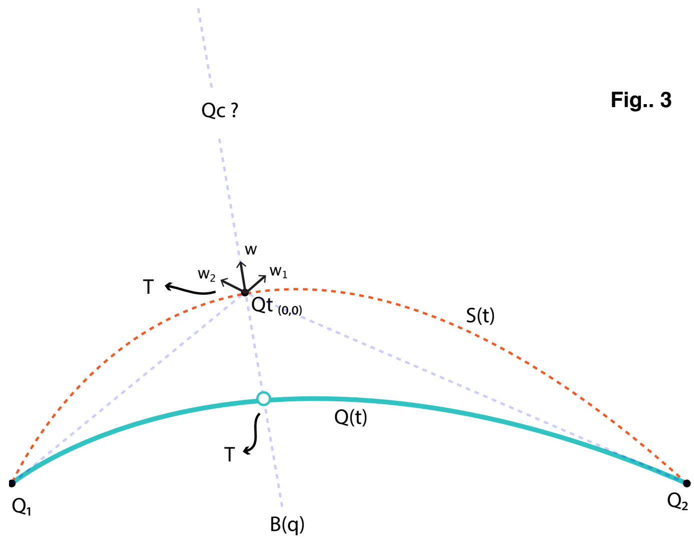
there is a P_c **"tension point"** that $P(t)$ pass through three points (P_1, P_t, P_2), where the point $P_t \in P(t)$ is one that the rect through P_c, P_t is one of the bisectors of P_1, P_t, P_2 through P_t (**Fig.2**) formed by the rect $A(r) = P_t + r\,m$ where

$m = m_1 + m_2$ and:

$$m_1 = \frac{P_1 - P_t}{|P_1 - P_t|} \qquad m_2 = \frac{P_2 - P_t}{|P_2 - P_t|}$$

DEMONSTRATION

Fig.. 3



Be the quadratic bezier that have the start and end control points P_1, P_2, and the "tension point" at P_t

$$P_t(t) = (1-t)^2 P_1 + 2t(1-t)P_t + t^2 P_2 \quad where \quad t \in [0,1]$$

Let's change the coordinate system (1) as $P_t \rightarrow (0,0)$ then

$$P_1 \rightarrow Q_1 = P_1 - P_t$$
$$P_t \rightarrow Q_t = (0,0)$$
$$P_2 \rightarrow Q_2 = P_2 - P_t$$
$$P_c \rightarrow Q_c = P_c - P_t$$
$$m \rightarrow w = w_1 + w_2 \text{ where } w_1 = \frac{Q_1}{|Q_1|}\ ; w_2 = \frac{Q_2}{|Q_2|}$$

So $\quad P_t(t) \rightarrow Q_t(t) = (1-t)^2 Q_1 + t^2 Q_2$

And the bisector rect through Q_t is $B(q) = q\,w$ where $q \in \mathbb{R}$

So the intersection between $Q(t)$ and $B(q)$ is

$$(1-t)^2 Q_1 + t^2 Q_2 = q\, w$$

So

$$\boxed{(1-t)^2} Q_1 + \boxed{t^2} Q_2 = q \left[\frac{Q_1}{|Q_1|} + \frac{Q_2}{|Q_2|} \right] = \boxed{\frac{q}{|Q_1|}} Q_1 + \boxed{\frac{q}{|Q_2|}} Q_2$$

Then

$$\frac{q}{|Q_1|} = (1-t)^2 \quad \text{and} \quad \frac{q}{|Q_2|} = t^2 \implies \underbrace{\frac{|Q_1|}{|Q_2|}}_{d^2} = \frac{t^2}{(1-t)^2} \implies \boxed{d = \frac{t}{1-t}}$$

So as $d > 0$ the solution for t is

$$\boxed{T = \frac{d}{d+1}}$$

Then by the “Equivalent quadratic beziers theorem” this value of T is the same on every intersection of that quadratic bezier that share the start and en points at Q_1, Q_2 and the “tension point” through the bisector, with the bisector. So is true particularly for the quadratic bezier

$$Q_c(t) = (1-t)^2 Q_1 + 2t(1-t) Q_c + t^2 Q_2$$

where

$$\boxed{Q_c(T) = Q_t = (0,0)} = (1-T)^2 Q_1 + 2T(1-T) Q_c + T^2 Q_2$$

So

$$Q_c = \frac{(1-T)^2 Q_1 + T^2 Q_2}{-2T(1-T)} = -\frac{1}{2} \left[\frac{1-T}{T} Q_1 + \frac{T}{1-T} Q_2 \right] = -\frac{1}{2} \left[\frac{1}{d} Q_1 + d Q_2 \right] =$$

$$= -\frac{1}{2} \left[\frac{\sqrt{|Q_2|}}{\sqrt{|Q_1|}} Q_1 + \frac{\sqrt{|Q_1|}}{\sqrt{|Q_2|}} Q_2 \right]$$

Then

$$\boxed{Q_c = -\frac{1}{2} \sqrt{|Q_1||Q_2|} \underbrace{\left[\frac{Q_1}{|Q_1|} + \frac{Q_2}{|Q_2|} \right]}_{w}}$$

So it's demonstrated Q_c and Q_t pass through the rect bisector $B(q)$ so in consequence for the coordinate translated quadratic bezier where P_c and P_t pass through one of the bisector of P_1, P_t, P_2 through P_t formed by the rect $A(r)$.

So

$$P_c = P_t - \frac{1}{2}\sqrt{|P_1 - P_t|\ |P_2 - P_t|}\left[\frac{P_1 - P_t}{|P_1 - P_t|} + \frac{P_2 - P_t}{|P_2 - P_t|}\right]$$

P_t is the near point of to P_c

There is other bisector perpendicular to m at P_t that is $n = m_1 - m_2$
this vector is parallel to the tangent at P_t of $P_c(t)$, as P_t and P_c pass through the bisector parallel to m then P_c is the near point to P_t as the tangent at this point is perpendicular to P_c

DEMONSTRATION

Playing on the coordinate system exposed at the before demonstration, and defining:

$$n \rightarrow u = w_1 - w_2$$

The tangent of Q_t is

$$Q_t' = Q_c'(T) = 2[(1-T)(Q_c - Q_1) + T(Q_2 - Q_c)]$$

with

$$T = \frac{d}{d+1} \quad \text{and} \quad Q_c = -\frac{1}{2}\left[\frac{1}{d}Q_1 + dQ_2\right]$$

then (after some boring operations)

$$Q_t' = dQ_2 - \frac{1}{d}Q_1$$

as the dot product:

$$Q_c \cdot Q_t' = 0$$

so when Q_t and Q_c pass through the rect bisector $B(q)$, then Q_c is perpendicular to the tangent at Q_t, so Q_t is the near point of $Q_c(t)$ to "tension point" Q_c.

In consequence translating back the coordinates: when P_t and P_c pass through the rect bisector $A(r)$, then P_c is perpendicular to the tangent at P_t, so P_t is the near point of $P_c(t)$ to the “tension point” P_c

Note: As the demonstration can be done in reverse is also true that the near point to Pc is the one that pass through the bisector.

References

1. *Algorithme de De Casteljau, P. de Casteljau. Outillages methodes calcul. Technical report, A Citroen, Paris 1959*
2. *Piecewise Linear Approximation of Bezier Curves Kaspar Fischer October 16, 2000*